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A solution to the problem of (A, B)-invariance for series

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Abstract

The aim of this article is to study a standard problem of control theory, the (A, B)-invariance problem, which amounts to computing a maximal element X subject to conditions of the form $AX \leqslant X + B$ and $X \leqslant K$. We give a solution to the problem in the framework of formal series over particular complete idempotent semirings. Over finite idempotent semirings, we show that, under the assumption that B and K are recognizable series, the maximal solution exists and is also recognizable. We obtain a similar result for the infinite tropical semiring, with additional hypothesis that the series A is a language, but the notion of recognizable series has to be extended to the weaker notion of pseudo-recognizable series. © 2002 Elsevier Science B.V. All rights reserved.

1. Introduction

The aim of this article is to prove that the supremum of the set

$$E = \{X \mid AX \leqslant X + B \text{ and } X \leqslant K\}, \tag{1}$$

where A is a language and B and K are pseudo-recognizable series on some complete idempotent semiring, is recognizable and computable when the semiring of coefficients is finite and pseudo-recognizable when the semiring of coefficients is the tropical semiring. The notion of pseudo-recognizability, defined in Section 5.2, generalizes the notion of recognizability. The order relation over series is specified in Section 4.1.

This problem is classical in control theory [8], in the setting of vector spaces, where A is a linear operator, X is a vector and B and K are vector spaces. We already solved it for recognizable languages [4], which are a particular case of recognizable series, by giving a constructive method to find the supremum.

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In [4], the solution was obtained by using a new operator called the left-cut and a new type of automata whose acceptance condition is a positive boolean formula. This solution is briefly sketched in Section 2. In Section 3, we set the framework of our problem: which semirings are considered and their basic properties. In Section 4, we introduce a new operator on formal series: the left-cut and we explore some of its properties. Then in Section 5, we introduce multi-representations, a new way to generate series, and we show how they are related to our problem. Note that in all these sections, we do not specialize our semiring: all results hold in any complete idempotent semiring in which addition distributes over infimum and vice-versa. In Section 6.1, we show that the supremum of the set E is in fact recognizable when the semiring of coefficients is finite and we prove the effective computability of the left-cuts and of the solution to our problem for such semirings. Finally, in Section 6.2, we prove that the supremum of the set E is pseudo-recognizable when the semiring of coefficients is the tropical semiring, A being a language.

2. Reviewing language theory

In this section, we briefly survey the theory of multi-automata and we see how to find the supremum of the set $\{X \mid AX \subseteq X \cup B \text{ and } X \subseteq K\}$, where A, B and K are fixed recognizable languages. For more details and proofs, the reader is referred to [4].

The left-cut of a language X by a language A is the language

$$A \backslash X = \{w \in \Sigma^* \mid Aw \subseteq X\}.$$

It is easy to see that $A\backslash X = \bigcap_{v \in A} v\backslash X$. If X is recognized by a complete deterministic automaton $(\Sigma, Q, i, \cdot, F)$, the language $v\backslash X$ is clearly recognized by the automaton $(\Sigma, Q, i \cdot v, \cdot, F)$. This remark leads directly to the following definition of a multi-automaton.

Definition 1. A *multi-automaton* is a structure $\mathscr{A} = (\mathbf{B}, \Phi)$, where

- $\mathbf{B}$ is a *transition system*, that is a triple $(\Sigma, Q, \cdot)$ such that $(q, x) \mapsto q \cdot x$ is a partial mapping from $Q \times \Sigma$ into Q; $\mathbf{B}$ is called the base of A.
- Φ is a positive boolean formula (i.e. without negation) on $Q \times \mathscr{P}(Q)$, called the *acceptance formula*.

We let $L^{\mathbf{B}}(j, F)$ be the language recognized by the automaton $(Q, j, \cdot, F)$. The language $L^{\mathbf{B}}(\Phi)$ *recognized* by a multi-automaton $(\mathbf{B}, \Phi)$ is the image of the boolean formula Φ by the morphism from the free distributive lattice over $Q \times \mathscr{P}(Q)$ into the distributive lattice $\mathscr{P}(\Sigma^*)$ obtained by mapping (j, F) to $L^{\mathbf{B}}(j, F)$.

In this article, we introduce a left-cut operator for series, which has the same basic properties as the left-cut operator for languages (see Lemmas 16 and 17). This leads to a new type of representations for series, which we call *multi-representations* (see Section 5.2).

In the context of languages [4], to find the supremum of the set

$$\{X \mid AX \subseteq X \cup B \text{ and } X \subseteq K\},$$

we introduced a sequence of languages defined by

$$K_0 = K \quad \text{and} \quad K_{n+1} = (A \backslash (B \cup K_n)) \cap K_n.$$

The element $K_\infty = \bigcap_n K_n$ was shown to be the solution of the problem and we gave a construction of a multi-automaton recognizing K_∞.

For formal series, the fundamental idea is the same. The main problem is that the semiring of coefficients may now not contain more than two elements. This will complicate the discussion, as we will see in Sections 6.1 and 6.2.

3. Idempotent semirings

In this section, we explore the basic properties of idempotent semirings and series.

3.1. Definition and basic properties

Definition 2. A *semiring* is a quintuple $(\mathscr{S}, +, *, 0, 1)$ with the following properties (see [2]):
- $(\mathscr{S}, +, 0)$ is a commutative monoid with identity element 0,
- $(\mathscr{S}, *, 1)$ is a monoid with identity element 1; as usual we denote by ab the product $a * b$ for all $a, b \in \mathscr{S}$,
- the element 0 is absorbing: $a0 = 0a = 0$ for all $a \in \mathscr{S}$,
- multiplication is distributive with respect to addition, i.e.: $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$, for all $a, b, c \in \mathscr{S}$.

A semiring is commutative if multiplication is commutative. It is *idempotent* if addition is idempotent.

We often write $(\mathscr{S}, +, *)$ or simply $\mathscr{S}$ for the semiring $(\mathscr{S}, +, *, 0, 1)$.

Examples 3.
- The boolean semiring $\mathbb{B} = \{0, 1\}$ is a finite commutative idempotent semiring.
- The tropical semiring $\mathbb{N}_{\min} = (\mathbb{N} \cup \{+\infty\}, \min, +, +\infty, 0)$ is an infinite commutative idempotent semiring.
- The set of (recognizable) languages over a fixed alphabet, with union for addition and concatenation for multiplication, is an infinite non-commutative idempotent semiring (with identity elements: for addition the empty set and for multiplication the singleton containing the empty word).

Throughout this paper, $\mathscr{S}$ denotes an idempotent semiring.

We consider the *natural order* over $\mathscr{S}$ given by $a \leqslant b$ if and only if there exists $c \in \mathscr{S}$ such that $b = a + c$.

Lemma 4. *For two elements a and b in $\mathscr{S}$, we have $a \leqslant b$ if and only if $b = a + b$.*

Proof. If $b = a+b$, then clearly $a \leqslant b$. Conversely, let us assume that $a \leqslant b$. There exists c such that $b = a + c$. Hence, we have $a + b = a + a + c = a + c = b$. $\square$

It follows in particular that the least element of $\mathscr{S}$ is 0.

Note that for the tropical semiring, the natural order is exactly the inverse of the usual order on $\mathbb{N}$, denoted by $\preccurlyeq$: in $\mathbb{N}$, we have $2 \preccurlyeq 3$, but in $\mathbb{N}_{\min}$, we have $3 \leqslant 2$.

Remark 5. Multiplication is compatible with the order. Indeed, let a, b and c be three elements of $\mathscr{S}$ and let us assume that $a \leqslant b$, i.e. $b = a + b$. Then $bc = (a+b)c = ac + bc$, that is $ac \leqslant bc$. In the same way, we obtain $ca \leqslant cb$.

3.2. Supremum and infimum

It is easy to see that if $\mathscr{T}$ is a non-empty finite subset of an idempotent semiring $\mathscr{S}$, the sum of its elements is its supremum. By analogy, if $\mathscr{T}$ is any subset of $\mathscr{S}$, we denote by $\sum_{x \in \mathscr{T}} x$ the supremum of $\mathscr{T}$, if it exists. This notation is justified since, in particular, the supremum of $\mathscr{T} \cup \mathscr{T}'$ is the sum of the suprema of $\mathscr{T}$ and $\mathscr{T}'$.

Recall that an ordered set is *complete* if each of its subsets has a supremum.

Definition 6. A semiring $\mathscr{S}$ is *complete* if it is complete as an ordered set and satisfies the following distributivity conditions:

$$\text{for all } \mathscr{T} \subseteq \mathscr{S} \text{ and all } s \in \mathscr{S}, \left(\sum_{t \in \mathscr{T}} t \right) s = \sum_{t \in \mathscr{T}} (ts) \text{ and } s \left(\sum_{t \in \mathscr{T}} t \right) = \sum_{t \in \mathscr{T}} (st).$$

Example 7. The tropical semiring is complete. Indeed, if $\mathscr{T}$ is a non-empty subset of $\mathbb{N} \cup \{+\infty\}$, either $\mathscr{T} = \{+\infty\}$ and $\sup \mathscr{T} = +\infty$, or $\mathscr{T}$ contains an integer and $\sup \mathscr{T}$ is the least integer of $\mathscr{T}$ for the usual order. The distributivity conditions are immediately verified.

We now suppose that $\mathscr{S}$ is a complete idempotent semiring.

In a complete idempotent semiring $\mathscr{S}$, every subset $\mathscr{T}$ has an infimum: the sum of all the elements x such that $x \leqslant t$ for each $t \in \mathscr{T}$. This infimum is written $\bigcap_{t \in \mathscr{T}} t$, or $a \cap b$ if $\mathscr{T} = \{a, b\}$. It follows directly from the definition that the operation of infimum is idempotent and compatible with the order.

Examples 8.

- If $\mathscr{S}$ is the set of languages on an alphabet Σ, the infimum of two languages is their intersection.
- If $\mathscr{S}$ is the tropical semiring $\mathbb{N}_{\min}$, the infimum of two elements is their maximum in the usual order.

Proposition 9. *In a complete idempotent semiring $\mathscr{S}$, the following distributivity property holds. Let Y be a subset of $\mathscr{S}$ and let $x \in \mathscr{S}$. Then*

$$x\left(\bigcap_{y\in Y} y\right) \leqslant \bigcap_{y\in Y}(xy) \quad \textit{and} \quad \left(\bigcap_{y\in Y} y\right)x \leqslant \bigcap_{y\in Y}(yx).$$

Proof. Both properties are clear, since for each $z \in Y$,

$$x\left(\bigcap_{y\in Y} y\right) \leqslant xz \quad \text{and} \quad \left(\bigcap_{y\in Y} y\right)x \leqslant zx. \qquad \square$$

In the rest of this paper, we suppose moreover that the operations $+$ and $\cap$ confer to $\mathscr{S}$ a structure of a distributive lattice, i.e. $+$ distributes over $\cap$ and $\cap$ over $+$. This property is used in Section 5.2 to define the series recognized by a multi-representation.

Examples 10.
- If $\mathscr{S}$ is the set of languages on an alphabet Σ, union and intersection distributes one over the other.
- If $\mathscr{S} = \mathbb{N}_{\min}$, minimum and maximum distributes one over the other (these are, respectively, the addition and infimum operations of the tropical semiring).

4. Series and left-cuts

The problem we want to solve, i.e. finding the supremum of the set

$$\{X \mid AX \leqslant X + B \text{ and } X \leqslant K\}$$

is stated in the framework of series. In order to keep this paper self-contained, the main definitions on series are recalled in Section 4.1. Furthermore, the solution found in [4] requires a new operator, the left-cut operator. In Section 4.2, we introduce a similar operator for series and extend the so-called pasting lemma.

4.1. Formal series

We consider the set $\mathscr{S}\langle\langle\Sigma\rangle\rangle$ of formal series on Σ, with coefficients in $\mathscr{S}$. A typical element of $\mathscr{S}\langle\langle\Sigma\rangle\rangle$ is written $A = \sum_{w\in\Sigma^*}(A,w)w$, with $(A,w) \in \mathscr{S}$ for each word w. The *support* of a series A, $\operatorname{Supp} A$, is the set of words w such that $(A,w) \neq 0$. The *image* of a series A, $\operatorname{Im} A$, is the set of coefficients of A. We identify an element a of $\mathscr{S}$ with the constant series, also denoted by a, defined by $(a,1) = a$ and $(a,w) = 0$ for every non-empty word w. In the same way, we identify a word $w \in \Sigma^*$ with the series also denoted by w and defined by $(w,u) = 0$ if $u \neq w$ and $(w,w) = 1$. Our notation is taken from [1].

A series is said to be a language if all its coefficients belong to $\{0,1\}$.

Operations on $\mathscr{S}$ are extended to the set of formal series by letting $(S+T,w)=(S,w)+(T,w)$ and $(ST,w)=\sum_{uv=u}(S,u)(T,v)$. These operations provide $\mathscr{S}\langle\langle\Sigma\rangle\rangle$ with a semiring structure. The natural order over $\mathscr{S}\langle\langle\Sigma\rangle\rangle$ is then exactly the extension of the order of $\mathscr{S}$: $X\leqslant Y$ if and only if for all $w\in\Sigma^*, (X,w)\leqslant(Y,w)$. Since $\mathscr{S}$ is idempotent and complete, $\mathscr{S}\langle\langle\Sigma\rangle\rangle$ is a complete idempotent semiring. In particular, the infimum $A\cap B$ of two elements $A,B\in\mathscr{S}\langle\langle\Sigma\rangle\rangle$ is given by

$$(A\cap B,w)=(A,w)\cap(B,w) \quad \text{for each } w\in\Sigma^*.$$

The distributivity of the lattice $(\mathscr{S}\langle\langle\Sigma\rangle\rangle,+,\cap)$ is inherited from the distributivity of the lattice $(\mathscr{S},+,\cap)$.

If A is a series and s an element of the semiring of coefficients $\mathscr{S}$, the *s-support* of A is the language $A^{-1}s=\{w\in\Sigma^*\mid(A,w)=s\}$.

Remark 11. If $\mathscr{S}=\mathbb{B}$, the set of series on the alphabet Σ, with coefficients in $\mathscr{S}$ can be identified with the set of languages on the alphabet Σ: $(A,w)=1$ if and only if $w\in A$. The order previously introduced corresponds to inclusion, the infimum on elements of $\mathscr{S}$ to conjunction and the infimum on series to intersection of languages.

4.2. *Left-cuts*

Let us go back a moment to languages. Let A and X be two languages, the notion of left-cut for languages was defined in [4, Definition 3.1] as follows:

$$A\backslash X=\{w\in\Sigma^*\mid Aw\subseteq X\}.$$

The left-cut enjoys the following property [4, Lemma 3.1]:

$$AX\subseteq Y \quad\Leftrightarrow\quad X\subseteq A\backslash Y.$$

If we look at A and X as two series over the boolean semiring, we can write

$$(A\backslash X,w)=\sum\{n\in\mathbb{B}\mid Awn\leqslant X\}.$$

Note that Awn is the series on $\mathscr{S}$ defined by $(Awn,u)=(A,v)n$, if there exists a word v such that $u=vw$, and $(Awn,u)=0$ if w is not a suffix of u.

In view of this remark, we define a left-cut for series as follows.

Definition 12. Let A and X be two series in $\mathscr{S}\langle\langle\Sigma\rangle\rangle$. The *left-cut* $A\backslash X$ of X by A is the series defined, for each word $w\in\Sigma^*$, by

$$(A\backslash X,w)=\sum_{n\in\mathscr{S}\mid Awn\leqslant X} n.$$

Observe that the set $\{n\in\mathscr{S}\mid Awn\leqslant X\}$ is not empty because it contains 0.

The left-cut has a characteristic property, that we first prove for elements of $\mathscr{S}$.

Lemma 13. *If a, x and y are elements of $\mathscr{S}$, we have*

$$ax \leqslant y \quad \Leftrightarrow \quad x \leqslant a \backslash y.$$

Proof. If $ax \leqslant y$, we have $x \leqslant \sum_{n \in \mathscr{S} | an \leqslant y} n = a \backslash y$. Conversely, if $x \leqslant a \backslash y$, $ax \leqslant a(a \backslash y) = a(\sum_{an \leqslant y} n) = \sum_{an \leqslant y} an \leqslant y$. $\square$

Thanks to this remark, we give an expression of the coefficients of $A \backslash X$ in terms of coefficients of A and X.

Proposition 14. *For each word $w \in \Sigma^*$,*

$$(A \backslash X, w) = \bigcap_{v \in \Sigma^*} ((A, v) \backslash (X, vw)). \tag{2}$$

Proof. Let p be the right-hand side of (2). We claim that $Awp \leqslant X$, that is, for all $u \in \Sigma^*$,

$$(Aw, u)p \leqslant (X, u).$$

This is trivial if w is not a suffix of u, since then, $(Aw, u) = 0$. Thus, we may assume that $u = sw$ for some $s \in \Sigma^*$. Then, since $p \leqslant (A, s) \backslash (X, sw)$, and since $(A, s) = (Aw, sw) = (A, u)$, we have

$$(Aw, u)p \leqslant (Aw, u)((Aw, u) \backslash (X, u)) \leqslant (X, u),$$

proving the claim. It follows that $p \leqslant (A \backslash X, w)$.

Conversely, let us prove that $(A \backslash X, w) \leqslant p$. It suffices to verify that if $Awn \leqslant X$, then for each $v \in \Sigma^*$,

$$n \leqslant (A, v) \backslash (X, vw).$$

But this follows from Lemma 13, since $(A, v)n = (An, v) = (Awn, vw) \leqslant (X, vw)$. $\square$

Remark 15. Eq. (2) is similar to the equation for the product of two series

$$(AX, w) = \sum_{vu = w} (A, v)(X, u) = \sum_{v \in \Sigma^*} (A, v)(X, v \backslash w),$$

where

$$v \backslash w = \{u \in \Sigma^* \mid \{v\}u \subseteq \{w\}\} = \{u \in \Sigma^* \mid vu = w\}.$$

There is a formal analogy between both formulas, replacing addition and multiplication, respectively, by infimum and left-cut.

We can now prove the basic property of the left-cut operator on series.

Lemma 16 (Pasting lemma for series). *If A, X and Y are series, the following equivalence holds*

$$AX \leqslant Y \quad \Leftrightarrow \quad X \leqslant A\backslash Y,$$

i.e. $A\backslash Y = \sup\{X \mid AX \leqslant Y\}$, *and in particular*

$$A(A\backslash Y) \leqslant Y \quad \text{and} \quad X \leqslant A\backslash(AX).$$

Proof. We first show that if $AX \leqslant Y$, then $X \leqslant A\backslash Y$. We need to show that, for every $w \in \Sigma^*$, $(X,w) \leqslant (A\backslash Y, w)$. For this, it is enough to prove that $Aw(X,w) \leqslant Y$, that is, for every $u \in \Sigma^*$

$$(Aw,u)(X,w) \leqslant (Y,u). \tag{3}$$

If w is not a suffix of u, then $(Aw,u)=0$ and the relation is trivial. Otherwise, $u=sw$ for some $s \in \Sigma^*$. Now, since $AX \leqslant Y$ by hypothesis,

$$(A,s)(X,w) \leqslant (AX,u) \leqslant (Y,u).$$

Relation (3) now follows from the observation that $(A,s)=(Aw,sw)=(Aw,u)$.

We now show that if $X \leqslant A\backslash Y$, then $AX \leqslant Y$. It is sufficient to prove that, for every s, $w \in \Sigma^*$,

$$(A,s)(X,w) \leqslant (Y,sw), \tag{4}$$

since $(AX,u) = \sum_{sw=u}(A,s)(X,w)$.

First, since $X \leqslant A\backslash Y$, $(X,w) \leqslant (A\backslash Y,w)$, whence

$$(A,s)(X,w) \leqslant (A,s)(A\backslash Y,w). \tag{5}$$

Next, by definition, $(A\backslash Y,w) = \sum_{Awn \leqslant Y} n$. Therefore

$$(A,s)(A\backslash Y,w) = \sum_{Awn \leqslant Y}(A,s)n = \sum_{Awn \leqslant Y}(Aw,sw)n \leqslant (Y,sw) \tag{6}$$

and the conjunction of Eqs. (5) and (6) gives Eq. (4). $\square$

We prove now some simple properties which will be used in the rest of the paper.

Lemma 17. *Let A, B, X and Y be series. We have*

$$A\backslash(B\backslash X) = (BA)\backslash X,$$

$$(A\backslash X) + (A\backslash Y) \leqslant A\backslash(X+Y)$$

and

$$(A\backslash X) \cap (A\backslash Y) = A\backslash(X \cap Y).$$

Proof. We first show that $A\backslash(B\backslash X) = (BA)\backslash X$. Let Y be a series. The result follows from the following sequence of equivalences:

$$Y \leqslant A\backslash(B\backslash X) \Leftrightarrow AY \leqslant B\backslash X \Leftrightarrow B(AY) \leqslant X$$
$$\Leftrightarrow (BA)Y \leqslant X \Leftrightarrow Y \leqslant (BA)\backslash X.$$

We now show that $(A\backslash X) + (A\backslash Y) \leqslant A\backslash(X+Y)$. We have $X \leqslant X+Y$, so $A\backslash X \leqslant A\backslash (X+Y)$ and in the same way $A\backslash Y \leqslant A\backslash(X+Y)$. Thus $(A\backslash X) + (A\backslash Y) \leqslant A\backslash(X+Y)$.

At last, we show that $(A\backslash X) \cap (A\backslash Y) = A\backslash(X \cap Y)$. Let Z be a series. We have

$$\begin{aligned} Z \leqslant A\backslash(X\cap Y) &\Leftrightarrow AZ \leqslant X \cap Y \\ &\Leftrightarrow AZ \leqslant X \text{ and } AZ \leqslant Y \\ &\Leftrightarrow Z \leqslant A\backslash X \text{ and } Z \leqslant A\backslash Y \\ &\Leftrightarrow Z \leqslant (A\backslash X) \cap (Z \leqslant A\backslash Y). \qquad \square \end{aligned}$$

Now we can build a sequence $(K_n)_{n\in\mathbb{N}}$ of series as follows:

$$K_0 = K,$$

$$\text{for } n \succeq 0,\ K_{n+1} = (A\backslash(B+K_n)) \cap K_n.$$

We call K_∞ the infimum of the sequence K_n:

$$K_\infty = \bigcap_{n \succeq 0} K_n. \tag{7}$$

Proposition 18. *The series K_∞ is the supremum of the set*

$$E = \{X \mid AX \leqslant X + B \text{ and } X \leqslant K\}.$$

Proof. Let us show that K_∞ is an element of E. First $K_\infty \leqslant K_0 = K$. Next, for all integers n,

$$\begin{aligned} AK_{n+1} &\leqslant A[A\backslash(Kn+B)] \\ &\leqslant K_n + B \quad \text{by the pasting lemma.} \end{aligned}$$

So by Proposition 9, for all n, $AK_\infty \leqslant K_n + B$. Taking the infimum on all n yields

$$AK_\infty \leqslant K_\infty + B.$$

Now we prove that K_∞ is the supremum of E. Let X be an arbitrary element of E. We show that $X \leqslant K_n$ by induction on $n \in \mathbb{N}$. For $n=0$, $X \leqslant K_0$ because $K_0 = K$ and X is an element of E. Let us assume that $X \leqslant K_n$. Then we have $AX \leqslant X + B \leqslant K_n + B$ and hence $X \leqslant A\backslash(K_n + B)$ by the pasting lemma. Since $X \leqslant K_n$ by hypothesis, it follows that $X \leqslant (A\backslash(K_n+B)) \cap K_n$, i.e. $X \leqslant K_{n+1}$. Taking the infimum over all n leads to $X \leqslant K_\infty$. $\square$

In the rest of the paper, we discuss methods to get some information on the series K_∞. In Section 5.3, we give a general result which allows us to effectively compute K_∞ when $\mathscr{S}$ is a finite semiring, and to prove that K_∞ is pseudo-recognizable when $\mathscr{S}$ is the infinite tropical semiring.

5. Where matrices interfere

5.1. Representations and recognizable series

Let $\mathscr{S}^{n\times n}$ denote the set of (n,m)-matrices with entries in $\mathscr{S}$. Recall [1, Chapter 1] that a series $S\in\mathscr{S}\langle\langle\Sigma\rangle\rangle$ is recognizable if and only if there exists an integer $n\geqslant 1$, a morphism of monoids

$$\mu : \Sigma^* \to \mathscr{S}^{n\times n}$$

and two matrices $\lambda\in\mathscr{S}^{1\times n}$ and $\gamma\in\mathscr{S}^{n\times 1}$ such that, for all words w

$$(S,w) = \lambda\mu(w)\gamma.$$

The triple (λ,μ,γ) is called a *linear representation* of S and n is its dimension.

A morphism $\mu:\Sigma^*\to\mathscr{S}^{n\times n}$ being fixed, we denote by $S(\lambda,\gamma)$ the series S defined by $(S,w)=\lambda\mu(w)\gamma$. The Kleene–Schützenberger theorem is the cornerstone of the theory of formal series: a formal series is rational if and only if it is recognizable (for more details, see [1]).

5.2. Multi-representations and pseudo-recognizable series

In this section, we extend the notion of linear representation introduced in Section 5.1. Our new notion is called multi-representation, in analogy with the multi-automata introduced in [4].

If c is an element of $\mathscr{S}$, we observe that the series $c\backslash S$ is defined by $(c\backslash S,w)=c\backslash(S,w)$. Given a triple $(c,\lambda,\gamma)\in\mathscr{S}\times\mathscr{S}^{1\times n}\times\mathscr{S}^{n\times 1}$, we denote by $S(c,\lambda,\gamma)$ the series S defined by $(S,w)=c\backslash(\lambda\mu(w)\gamma)$.

Let n be an integer, μ be a morphism of monoids $\Sigma^*\to\mathscr{S}^{n\times n}$ and Φ be a positive boolean formula on $\mathscr{S}\times\mathscr{S}^{1\times n}\times\mathscr{S}^{n\times 1}$. It is convenient to call *atom* an element of $\mathscr{S}\times\mathscr{S}^{1\times n}\times\mathscr{S}^{n\times 1}$.

Definition 19. The series $S(\Phi)$ is the image of Φ by the morphism from the free distributive lattice over $\mathscr{S}\times\mathscr{S}^{1\times n}\times\mathscr{S}^{n\times 1}$ into the distributive latticc $(\mathscr{S}\langle\langle\Sigma\rangle\rangle,+,\cap)$ obtained by mapping (c,λ,γ) to $S(c,\lambda,\gamma)$.

The pair (μ,Φ) is by definition a *multi-representation* of S. By analogy with languages and automata, we say that μ is the *base* and Φ the *acceptance formula* of the multi-representation (μ,Φ). If the base μ is fixed, we say (improperly) that Φ is the acceptance formula of the series S. The series S is recognized by (μ,Φ). If an atom (c,λ,γ) occurs in the acceptance formula of a multi-representation of S, c is a coefficient, λ an *initial vector* and γ a *final vector* of the multi-representation of S.

Two representations are said to be *equivalent* if they represent the same series.

Definition 20. A series is *pseudo-recognizable* if it has a multi-representation.

As it is shown in the next example, pseudo-recognizable series are not necessarily recognizable.

Example 21. Let $\Sigma = \{a, b\}$ and $\Sigma' = \{c\}$. We consider series with coefficients in the commutative semiring of languages on the alphabet Σ' (sum is union and product is concatenation of sets).

Let S and T be the series defined as follows: for each word $w \in \Sigma^*$,

$$(S, w) = \{c^{|w|_a}\} \quad \text{and} \quad (T, w) = \{c^{|w|_b}\},$$

where $|w|_x$ denotes the number of occurrences of the letter x in w.

Both series are recognizable and admit a linear representation with the same base. Indeed, let $\mu : \Sigma^* \to \mathscr{S}^{2\times 2}$ be the morphism defined by

$$\mu(a) = \begin{pmatrix} \{c\} & \emptyset \\ \emptyset & \{1\} \end{pmatrix} \quad \text{and} \quad \mu(b) = \begin{pmatrix} \{1\} & \emptyset \\ \emptyset & \{c\} \end{pmatrix}$$

and let $\lambda_S = (\{1\}\ \emptyset)$, $\gamma_S = \left(\begin{smallmatrix} \{1\} \\ \emptyset \end{smallmatrix}\right)$, $\lambda_T = (\emptyset\ \{1\})$ and $\gamma_T = \left(\begin{smallmatrix} \emptyset \\ \{1\} \end{smallmatrix}\right)$. Then $(S, w) = \lambda_S \mu(w) \gamma_S$ and $(T, w) = \lambda_T \mu(w) \gamma_T$.

Now, the infimum of these series is pseudo-recognizable. The coefficient of a word w in $S \cap T$ is the intersection of (S, w) and (T, w) (see Example 8). So we have

$$(S \cap T, w) = \begin{cases} \emptyset & \text{if } |w|_a \neq |w|_b, \\ c^{|w|/2} & \text{if } |w|_a = |w|_b (= \frac{|w|}{2}). \end{cases}$$

But Pin and Sakarovitch have shown in [6] that the inverse image of a recognizable language by a recognizable transduction is recognizable. Since the support of a transduction $\Sigma^* \to \Sigma'^*$ is the inverse image of Σ'^*, the series $S \cap T$ is pseudo-recognizable and not recognizable.

We will see in Section 6.1 that if the semiring of coefficients is finite, then a formal series is pseudo-recognizable if and only if it is recognizable.

Remark 22. In general, there exist series which are not pseudo-recognizable. For example, if the semiring of coefficients is countable, there is a countable number of pseudo-recognizable series, but the set of series is not countable.

Example 23. It is easy to see, as a special case of Corollary 31, that the pseudo-recognizable series with coefficients in $\mathbb{B}$ are recognizable. Indeed, these series can be identified with their supports and the infimum of two series is their intersection. In particular, the series $\sum_{n\in\mathbb{N}} a^n b^n$ is not pseudo-recognizable in $\mathbb{B}\langle\langle\Sigma\rangle\rangle$.

Lemma 24. *Any two pseudo-recognizable series admit multi-representations with the same base.*

Proof. Let A and B be two pseudo-recognizable series given by multi-representations, respectively, a morphism $\mu_A : \Sigma^* \to \mathscr{S}^{n_A \times n_A}$ and a positive boolean formula Φ_A on $\mathscr{S} \times \mathscr{S}^{1\times n_A} \times \mathscr{S}^{n_A \times 1}$, and a morphism $\mu_B : \Sigma^* \to \mathscr{S}^{n_B \times n_B}$ and a positive boolean formula Φ_B on $\mathscr{S} \times \mathscr{S}^{1\times n_B} \times \mathscr{S}^{n_B \times 1}$. We build new multi-representations as follows. Let $n = n_A + n_B$, and let $\mu : \Sigma^* \to \mathscr{S}^{n\times n}$ given by

$$\mu = \left(\begin{array}{c|c} \mu_A & 0 \\ \hline 0 & \mu_B \end{array}\right).$$

We build the new acceptance formula Φ'_A for A by replacing the row vectors $\lambda \in \mathscr{S}^{1\times n_A}$ and column vectors $\gamma \in \mathscr{S}^{n_A\times 1}$ which occur in Φ_A, respectively, by

$$\lambda' = (\lambda|0) \in \mathscr{S}^{1\times n} \quad \text{and} \quad \gamma' = \left(\begin{array}{c} \gamma \\ \hline 0 \end{array}\right) \in \mathscr{S}^{n\times 1}$$

and the new acceptance formula Φ'_B for B by replacing the $\lambda \in \mathscr{S}^{1\times n_B}$ and the $\gamma \in \mathscr{S}^{n_B\times 1}$ which occur in Φ_B, respectively, by

$$\lambda' = (0|\lambda) \in \mathscr{S}^{1\times n} \quad \text{and} \quad \gamma' = \left(\begin{array}{c} 0 \\ \hline \gamma \end{array}\right) \in \mathscr{S}^{n\times 1}.$$

Let us prove that (μ, Φ'_A) recognizes the series A. Let w be an arbitrary word. Let $\lambda \in \mathscr{S}^{1\times n_A}$ and $\gamma \in \mathscr{S}^{n_A\times 1}$ be two vectors which occur in Φ_A. We build $\lambda' \in \mathscr{S}^{1\times n}$ and $\gamma' \in \mathscr{S}^{n\times 1}$ as previously. It is enough to prove that $\lambda\mu_A(w)\gamma = \lambda'\mu(w)\gamma'$. We have

$$\begin{aligned} \lambda'\mu(w)\gamma' &= (\lambda|0) \left(\begin{array}{c|c} \mu_A(w) & 0 \\ \hline 0 & \mu_B(w) \end{array}\right) \left(\begin{array}{c} \gamma \\ \hline 0 \end{array}\right) \\ &= (\lambda|0) \left(\begin{array}{c} \mu_A(w)\gamma \\ \hline 0 \end{array}\right) = \lambda\mu_A(w)\gamma. \end{aligned}$$

In the same way, we prove that (μ, Φ'_B) recognizes the series B. □

Corollary 25. *The set of pseudo-recognizable series with a given base is closed under addition and infimum.*

5.3. Towards a multi-representation for the left-cut

In this section, we "almost" obtain a multi-representation for the series $A\backslash X$, where A is an arbitrary series and X a series given by a multi-representation (μ, Φ). We show in Section 6.1 that the supremum of the set $E = \{X \mid AX \leqslant X + B \text{ and } X \leqslant K\}$ is recognizable and we explain how to compute a multi-representation when A is a recognizable series, when the semiring $\mathscr{S}$ is finite.

In Section 6.2, we use the result of this section to prove that the supremum of the set E is pseudo-recognizable, when $\mathscr{S}$ is the tropical semiring.

Proposition 26. *Let A be a series and let X be a pseudo-recognizable series given by a multi-representation (μ, Φ). Then*

$$A\backslash X = \bigcap_{v \in \Sigma^*} S(r_v(\Phi)), \tag{8}$$

where r_v is the endomorphism of the free distributive lattice over $\mathscr{S} \times \mathscr{S}^{1\times n} \times \mathscr{S}^{n\times 1}$ defined by $r_v(c, \lambda, \gamma) = (c(A, u), \lambda\mu(v), \gamma)$.

Proof. We denote by B the right-hand side of Formula (8).

We write Φ in normal form: there exists P_X, a finite set of subsets of $\mathscr{S} \times \mathscr{S}^{1\times n} \times \mathscr{S}^{n\times 1}$ such that

$$\Phi = \bigvee_{E \in P_X} \bigwedge_{(c,\lambda,\gamma)\in E} S(c, \lambda, \gamma).$$

By Lemma 17, we know that for a word u

$$(A, v)\backslash(c\backslash(\lambda\mu(\mu)\gamma)) = (c(A, v))\backslash(\lambda\mu(u)\gamma).$$

Let s be an element of $\mathscr{S}$. We have

$$\begin{aligned}
s \leqslant (A\backslash X, w) &\Leftrightarrow s \leqslant \bigcap_{v\in\Sigma^*} (A, v)\backslash(X, vw)\\
&\Leftrightarrow \forall v \in \Sigma^*, s \leqslant (A, v)\backslash(X, vw)\\
&\Leftrightarrow \forall v \in \Sigma^*, (A, v)s \leqslant (X, vw)\\
&\Leftrightarrow \forall v \in \Sigma^*, \exists E \in P_X, (A, v)s \leqslant \bigcap_{(c,\lambda,\gamma)\in E} c\backslash(\lambda\mu(vw)\gamma)\\
&\Leftrightarrow \forall v \in \Sigma^*, \exists E \in P_X, \forall(c, \lambda, \gamma) \in E, (A, v)s \leqslant c\backslash(\lambda\mu(vw)\gamma)\\
&\Leftrightarrow \forall v \in \Sigma^*, \exists E \in P_X, \forall(c, \lambda, \gamma) \in E, s \leqslant (c(A, v))\backslash(\lambda\mu(vw)\gamma)\\
&\Leftrightarrow s \leqslant (B, w)
\end{aligned}$$

and therefore,

$$A\backslash X = \bigcap_{v\in\Sigma^*} \sum_{E\in P_X} \bigcap_{(c,\lambda,\gamma)\in E} S(c(A, v), \lambda\mu(v), \gamma),$$

which is equivalent to Eq. (8). $\square$

This result "almost" gives formulas for multi-representations of a left-cut. These are not proper multi-representations since a boolean formula may not have a conjunction over an infinite set. However, we use these "almost" formulas in Sections 6.1 and 6.2 to answer the original question of this article.

6. Some particular semirings

With Eq. (8) in hand, we have an explicit formula to compute left-cuts of pseudo-recognizable series. However, this formula does not give, in general, an effective algorithm. The aim of this section is to show that, for some semirings, this equation gives results about the recognizability or the pseudo-recognizability of K_∞ (respectively, for finite semirings and for the tropical semiring), and even a constructive way to find K_∞ for finite tropical semirings.

6.1. Finite semirings

In this section, the semiring $\mathscr{S}$ of coefficients is supposed to be finite. We first show that in this case a formal series is pseudo-recognizable if and only if it is recognizable (Corollary 31). We show actually a stronger result: given a multi-representation, one can effectively compute an equivalent linear representation. Finally, we prove that the series K_∞ is recognizable.

Examples 27.

- The boolean semiring is finite.
- The *finite tropical semirings* are the quotient of the tropical semiring $\mathbb{N}_{\min}$, and can be defined as follows. For an integer $r \geqslant 1$, consider the finite semiring $\mathbb{N}_r = \{0, 1, \ldots, r\} \cup \{+\infty\}$, with min as addition and an r-"threshold" addition as multiplication, given by $xy = \min(x + y, r)$.

Our proof that a pseudo-recognizable series is in fact recognizable relies on a characterization of recognizability for series with coefficients in a finite semiring.

Proposition 28. *A series is recognizable if and only if, for each $s \in \mathscr{S}$, its s-support is a rational language.*

Proof. Let $A \in \mathscr{S}\langle\langle\Sigma\rangle\rangle$ be recognizable and let $s \in \mathscr{S}$. Then it is shown in [1, Proposition III.2.3], that $A^{-1}s$ is rational for every $s \in \mathscr{S}$. Conversely, if this condition holds, then by [1, Proposition III.2.1], the series $\sum_{w \in A^{-1}s} w$ is recognizable. The result follows since

$$A = \sum_{s \in \mathscr{S}} s \left(\sum_{w \in A^{-1}s} w \right). \qquad \square$$

Corollary 29. *If S_1 and S_2 are recognizable series over $\mathscr{S}$, then so is $S_1 \cap S_2$.*

Proof. Let $s \in \mathscr{S}$. Then

$$\begin{aligned}(S_1 \cap S_2)^{-1}s &= \{w \in \Sigma^* \mid (S_1 \cap S_2, w) = s\}\\ &= \left(S_1^{-1}s \cap \left(\bigcap_{t \geqslant s} S_2^{-1}t\right)\right) \cup \left(\left(\bigcap_{t \geqslant s} S_1^{-1}t\right) \cap S_2^{-1}s\right). \end{aligned} \tag{9}$$

Now, since S_1 and S_2 are recognizable, the sets of the form $S_1^{-1}s$ or $S_2^{-1}s$ are rational. Since rational sets are closed under intersection and union, $(S_1 \cap S_2)^{-1}s$ is also rational, and thus, by Proposition 28, $S_1 \cap S_2$ is recognizable. $\square$

Proposition 30. *If $c \in \mathscr{S}$ and S is a recognizable series over $\mathscr{S}$, then $c\backslash S$ is recognizable.*

Proof. We claim that

$$c\backslash S = \sum_{s\in\mathscr{S}} (c\backslash s)S^{-1}s.$$

Indeed, for each word $w \in \Sigma^*$ and every $s \in \mathscr{S}$, we have $(S^{-1}s, w) = 1$ if $(S, w) = s$ and $(S^{-1}s, w) = 0$ if $(S, w) \neq s$. So

$$\sum_{s\in\mathscr{S}} (c\backslash s)(S^{-1}s, w) = \sum_{s\in\mathscr{S}} \sum_{w\in S^{-1}s} c\backslash s = \sum_{s\in\mathscr{S}} c\backslash(S, w) = \sum_{s\in\mathscr{S}} (c\backslash S, w). \qquad \square$$

Corollary 31. *If $\mathscr{S}$ is a finite idempotent semiring, every pseudo-recognizable series is recognizable.*

Corollary 32. *Let X be a recognizable series. Then the series $A\backslash X$ is recognizable for any series A.*

Proof. Let n be the dimension of a multi-representation of X.

Since $\mathscr{S}$ is finite, the set of endomorphisms of the distributive lattice over $\mathscr{S} \times \mathscr{S}^{1\times n} \times \mathscr{S}^{n\times 1}$ is also finite. Therefore, the infimum defining $A\backslash X$ in Formula (8) can be replaced by a finite infimum. Hence, the series $A\backslash X$ is pseudo-recognizable, and so it is recognizable by Corollary 31. $\square$

Now that we have seen that the set of recognizable series is closed under left-cut, we prove that the left-cut is computable.

Proposition 33. *Given a linear representation of a recognizable series, one can effectively compute its s-supports.*

Proof. Let $s \in \mathscr{S}$ and let A be a recognizable series with (λ, μ, γ) as linear representation. We have

$$\begin{aligned} A^{-1}s &= \{w \in \Sigma^* \mid \lambda\mu(w)\gamma = s\} \\ &= \left\{ w \in \Sigma^* \;\middle|\; \sum_{i,j} \lambda_i \mu(w)_{i,j}\gamma_j = s \right\}. \end{aligned}$$

Let $P = \{m \in \mathscr{S}^{n\times n} \mid \sum_{i,j} \lambda_i m_{i,j}\gamma_j = s\}$. Then $A^{-1}s = \{w \in \Sigma^* \mid \mu(w) \in P\} = \mu^{-1}(P)$. Since $\mathscr{S}^{n\times n}$ is a finite monoid, such languages are effectively computable [5]. $\square$

This proposition is not sufficient: we know that a pseudo-recognizable series is in fact recognizable, but we do not know yet how to obtain a linear representation from a multi-representation. The next proposition solves this problem.

Proposition 34. *The s-supports of a recognizable series* given by a multi-representation *are effectively computable.*

Proof. Let $A = S(\Phi)$ be a recognizable series. If $\Phi = (1, \lambda, \gamma)$, we are in the case of a linear representation and the problem is solved by Proposition 33. If $\Phi = (c, \lambda, \gamma)$, we have seen that for a word w

$$(A, w) = \sum_{s \in \mathscr{S}} (c \backslash s) S^{-1} s,$$

where $S = S(\lambda, \gamma)$ (Proposition 30). Since $\mathscr{S}$ is finite and the s-supports of S are computable, so are the s-supports of X.

From Eq. (9), it is clear that if the s-supports of two series S_1 and S_2 are computable, so are the s-supports of $(S_1 \cap S_2)$. And we obtain the same result for $(S_1 + S_2)$ by observing that

$$\begin{aligned}(S_1 + S_2)^{-1} s &= \{w \in \Sigma^* \mid (S_1 + S_2, w) = s\} \\ &= \bigcap_{t_1 t_2 = s} (S_1^{-1} t_1 \cap S_2^{-1} t_2). \qquad \square\end{aligned}$$

Given a multi-representation of a series S, we obtain an equivalent linear representation considering that $S = \sum_{s \in \mathscr{S}} s S^{-1} s$.

Recall that the Hadamard product of two series A and X is the series $A \odot B$ such that for all word w, $(A \odot B, w) = (A, w)(B, w)$. We extend this notion, identifying a language to a series: if A is a language and B a formal series, we denote by $A \odot B$ the series $\sum_{w \in A} (B, w) w$.

Proposition 35. *Given a rational language A and a recognizable series B, one can effectively compute a linear representation for their Hadamard product.*

Proof. Recall that the tensor product of two matrices $M \in \mathscr{S}^{l \times m}$ and $N \in \mathscr{S}^{n \times p}$ is the matrix $M \otimes N \in \mathscr{S}^{ln \times mp}$ such that

$$M \otimes N = \begin{pmatrix} M_{1,1} N & \cdots & M_{1,m} N \\ \vdots & \ddots & \vdots \\ M_{l,1} N & \cdots & M_{l,m} N \end{pmatrix}.$$

We consider A like a series over $\mathbb{B}$. Since A is recognizable, it has a linear representation of dimension n', say $(\lambda', \mu', \gamma')$. Let $(\lambda'', \mu'', \gamma'')$ be a linear representation of dimension n'' of B. We let $n = n'n''$, $\lambda = \lambda' \otimes \lambda''$, $\mu = \mu' \otimes \mu''$ and $\gamma = \gamma' \otimes \gamma''$. Let $w \in \Sigma^*$ be any word. We show that (λ, μ, γ) is a linear representation of $A \otimes B$. We have

$$\begin{aligned}\lambda \mu(w) \gamma &= (\lambda' \otimes \lambda'')((\mu' \otimes \mu'')(w))(\gamma' \otimes \gamma'') \\ &= \sum_{1 \leqslant k,\, l \leqslant n'n''} (\lambda' \otimes \lambda'')_k ((\mu' \otimes \mu'')(w))_{k,l} (\gamma' \otimes \gamma'')_l\end{aligned}$$

$$= \sum_{\substack{1\leqslant i,i'\leqslant n' \\ 1\leqslant j,j'\leqslant n''}} \lambda'_i \lambda''_j (\mu'(w))_{i,i'} (\mu''(w))_{j,j'} \gamma'_{i'} \gamma''_{j'}.$$

Now, the elements of $\mathbb{B}$ commute with the elements of $\mathscr{S}$, so we have

$$\begin{aligned}
\lambda\mu(w)\gamma &= \sum_{\substack{1\leqslant i,i'\leqslant n' \\ 1\leqslant j,j'\leqslant n''}} \lambda'_i (\mu'(w))_{i,i'} \gamma'_{i'} \lambda''_j (\mu''(w))_{j,j'} \gamma''_{j'} \\
&= \left(\sum_{1\leqslant i,i'\leqslant n'} \lambda'_i (\mu'(w))_{i,i'} \gamma'_{i'} \right) \left(\sum_{1\leqslant j,j'\leqslant n''} \lambda''_j (\mu''(w))_{j,j'} \gamma''_{j'} \right) \\
&= \lambda' \mu'(w) \gamma' \cdot \lambda'' \mu''(w) \gamma'' \\
&= (A, w)(B, w) \\
&= (A \otimes B, w). \qquad \square
\end{aligned}$$

We now come back to Formula (8). We would like to replace the intersection over $v \in \Sigma^*$ by some finite intersection. The idea is the following: we order the atoms which appear in the acceptance formula of the multi-representation of X, say $\lambda_1, \ldots, \lambda_m$, and we look at the possible values of the row vectors $((A, v), \lambda_1\mu(v), \ldots, \lambda_m\mu(v))$. Note that the set of words v for which this vector is equal to $(s, t_{11}, \ldots, t_{1n}, t_{21}, \ldots, t_{mn})$ is exactly

$$(\ldots((A^{-1}s \odot X_{11})^{-1}t_{11} \odot X_{21})^{-1}t_{12}\ldots)^{-1}t_{m(n-1)} \odot X_{nm})^{-1}t_{mn}, \tag{10}$$

where X_{ij} is the series $S(\lambda_i, \varepsilon_j)$, with $\varepsilon_j = (0, \ldots, 0, 1, 0, \ldots, 0)$, where the unique 1 lies in position j. The coefficient (X_{ij}, v) is then equal to the jth coefficient of the row vector $\lambda_i\mu(v)$.

Now, we have seen that the Hadamard product of a language and a pseudo-recognizable series is computable (Proposition 35) and that the s-supports of a pseudo-recognizable series are constructible (Proposition 34), so the language defined by Formula (10) is effectively computable.

Fig. 1 gives an idea how to compute the left-cut, when X is given by a linear representation (X_i means X_{1i}). Indeed, if for example $A^{-1}s$ is an empty set, it is not useful to continue the computation of Formula (10), we know that the language it computes is empty. Generalization to the case of a multi-representation is immediate.

Now that we know how to compute a left-cut, we can come back to the original problem, that is finding the supremum of $\{X \mid AX \leqslant X + B \text{ and } X \leqslant K\}$.

We suppose that we have multi-representations for B and K, which have the same base μ (it is possible thanks to Lemma 24). We denote by Φ_B the acceptance formula of B and by Φ_K the acceptance formula of K.

Let us return to our sequence of series $(K_n)_{n\in\mathbb{N}}$.

We call *adequate* a multi-representation with base μ and such that the final vectors which appear in its acceptance formula are among those which appear in Φ_B or in Φ_K.

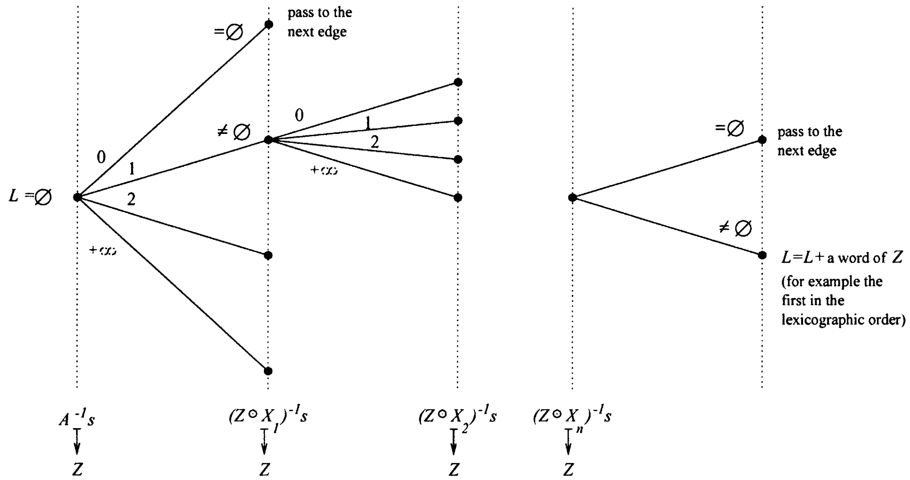

Fig. 1. An idea of the algorithm to compute a left-cut when $\mathscr{S}=\mathbb{N}_2$.

The series $K_0=K$ is recognized by an adequate multi-representation, and $B+K_0$ also: we take the disjunction of the acceptance formulas for K and for B. We have seen in Proposition 26 that $A\backslash(B+K_0)$ is then also recognized by an adequate multi-representation, since we assume $\mathscr{S}$ is finite. The series K_1 is then recognized by an adequate multi-representation: we take the conjunction of the acceptance formula for $A\backslash(B+K_0)$ and the one for K. By iteration, for each integer n, we can build an adequate multi-representation which recognizes K_n.

Since the semiring $\mathscr{S}$ is finite, so is the set $\mathscr{S}\times\mathscr{S}^{1\times n}\times\mathscr{S}^{n\times 1}$, and we will stump on a previously met multi-representation. But the sequence $(K_n)_{n\in\mathbb{N}}$ is decreasing by construction, so it is ultimately constant.

6.2. The tropical semiring

In this section, to simplify notations, we shall denote by $+\infty$ the neutral element for addition and by 0 the neutral element for multiplication. Unless otherwise indicated, the order used in this section is the natural order induced by the tropical structure: $+\infty\leqslant\cdots\leqslant n\leqslant\cdots\leqslant 1\leqslant 0$. The notation $\leqslant$-maximum (or $\leqslant$-minimum) always refers to the order $\leqslant$. Recall that $\preccurlyeq$ denotes the usual order on the integers, so that $2\preccurlyeq 3$.

We extend the usual substraction on integers by setting $(+\infty)-(+\infty)=0$ and $(+\infty)-c=+\infty$ if $c\neq+\infty$. The next lemma shows that left-cuts play the role of substraction in the tropical semiring.

Lemma 36. *Let* $c,d\in\mathbb{N}_{\min}$. *Then*

$$c\backslash d=\begin{cases} d-c & \textit{if } c \textit{ is smaller than } d \textit{ for the usual order},\\ 0 & \textit{otherwise.}\end{cases}$$

Proof. It is an immediate consequence of the definition and of Lemma 13. □

Corollary 37. *Let* $c, d, e \in \mathbb{N}_{\min}$. *Then* $(c\backslash d) + (c\backslash e) = c\backslash(d + e)$.

We now see a specific property of multi-representations over the tropical semiring, which leads to major simplifications. We prove that any pseudo-recognizable series admits a multi-representation with a unique final vector, a unique coefficient and whose acceptance formula does not contain any disjunctions. Note that the result is trivial if the semiring is finite because the series is then recognizable (Corollary 31).

Lemma 38. *If* X *is a pseudo-recognizable series, then there exists a multi-representation of* X, *which has only one final vector and one unique coefficient.*

Proof. Let Φ be a multi-representation of X. Let $(c_1, \lambda_1, \gamma_1), \ldots, (c_m, \lambda_m, \gamma_m)$ be the atoms of Φ, which recognize, respectively, the series $X_1, \ldots, X_m$. We may suppose that $c_1 \geqslant \cdots \geqslant c_m$, so there exist $s_1, \ldots, s_{m-1} \in \mathbb{N}_{\min}$ such that $c_m = s_i c_i$ for $i \in \{1, \ldots, m\}$, with $s_m = 1$. We construct the following vectors and matrices, for $i \in \{1, \ldots, m\}$:

$$\lambda'_i = (+\infty| \ldots | +\infty | s_i\lambda_i | +\infty | \ldots | +\infty),$$

$$\mu' = I_n \otimes \mu = \left(\begin{array}{c|c|c} \mu & +\infty & +\infty \\ \hline +\infty & \ddots & +\infty \\ \hline +\infty & +\infty & \mu \end{array}\right),$$

$$\gamma' = \left(\begin{array}{c} \gamma_1 \\ \hline \vdots \\ \hline \gamma_m \end{array}\right).$$

We claim that $X_i = S^{\mu'}(c_m, \lambda'_i, \gamma')$. Indeed, we have, for any word w:

$$\begin{aligned} \lambda'_i \mu'(w)\gamma' &= (+\infty| \ldots | s_i\lambda_i | \ldots | +\infty) \left(\begin{array}{c|c|c} \mu & +\infty & +\infty \\ \hline +\infty & \ddots & +\infty \\ \hline +\infty & +\infty & \mu \end{array}\right) \left(\begin{array}{c} \gamma_1 \\ \hline \vdots \\ \hline \gamma_m \end{array}\right) \\ &= s_i\lambda_i\mu(w)\gamma_i. \end{aligned}$$

So

$$(X_i, w) = c_i\backslash(\lambda_i\mu(w)\gamma_i) = (s_ic_i)\backslash(s_i\lambda_i\mu(w)\gamma_i) = c_m\backslash(\lambda'_i\mu'(w)\gamma'). \qquad \square$$

Proposition 39. *Any pseudo-recognizable series over a tropical semiring admits a multi-representation with a unique final vector, a unique coefficient and whose acceptance formula does not contain any disjunctions. Moreover, such a multi-representation is computable.*

Proof. Let X be a series over a tropical semiring and let (μ, Φ) be a multi-representation of X with a unique final vector and a unique coefficient (it is computable according to Lemma 38). Let w be a word. Assume that atoms (c, λ_1, γ) and (c, λ_2, γ) appears in Φ. By Corollary 37, we have $(c\backslash(\lambda_1\mu(w)\gamma)) + (c\backslash(\lambda_2\mu(w)\gamma)) = c\backslash((\lambda_1 + \lambda_2)\mu(w)\gamma)$. $\square$

In this section, we prove that the supremum element of the set $E = \{X \mid AX \leqslant X + B$ and $X \leqslant K\}$ is pseudo-recognizable if B and K are pseudo-recognizable and A is a language. The solution is unfortunately not constructive. We first show that each term of the sequence $(K_n)_{n\in\mathbb{N}}$ is pseudo-recognizable and then that the limit K_∞ is pseudo-recognizable. Note that in this section, we do not need the assumption that A is pseudo-recognizable: it could be any language.

6.2.1. The left-cut by a series of finite image is a pseudo-recognizable operation

Theorem 40. *If the series X is pseudo-recognizable and the series A has a finite image, the series $A\backslash X$ is pseudo-recognizable. More precisely, if X is given by a multi-representation with a unique final vector, a unique coefficient c and an acceptance formula without disjunction, then there exists a multi-representation of $A\backslash X$ with the same base, the same unique final vector, an acceptance formula without any disjunction and such that its unique coefficient is equal to cs_0 where $s_0 = \bigcap_{s\in Im\, A-\{+\infty\}} s$.*

To prove this theorem, we need the following lemmas. Lemmas 41 and 42 explain how to eliminate words of the infimum of Eq. (8). Lemma 46 shows that these procedures indeed yield finite sets.

Let $n \geqslant 1$ be an integer, $\mu : \Sigma^* \to \mathbb{N}_{\min}^{n\times n}$ be a morphism and $\lambda \in \mathbb{N}_{\min}^{1\times n}$ and $\gamma \in \mathbb{N}_{\min}^{n\times 1}$ be two vectors.

Lemma 41 (First elimination lemma). *Let v_1 and v_2 be two words such that $(A, v_1) \geqslant (A, v_2)$ and $\lambda\mu(v_1) \leqslant \lambda\mu(v_2)$, then*

$$S(c(A, v_1), \lambda\mu(v_1), \gamma) \leqslant S(c(A, v_2), \lambda\mu(v_2), \gamma).$$

Proof. Let w be any word. For each $k \in \{1, \ldots, n\}$, we have $(\lambda\mu(v_1))_k \leqslant (\lambda\mu(v_2))_k$, so $(\lambda\mu(v_1))_k(\mu(w)\gamma)_k \leqslant (\lambda\mu(v_2))_k(\mu(w)\gamma)_k$, and summing on all $k \in \{1, \ldots, n\}$ yields $S(\lambda\mu(v_1), \gamma) \leqslant S(\lambda\mu(v_2), \gamma)$.

Now, we have seen in Lemma 36 that cutting by an element of $\mathbb{N}_{\min}$ is like taking away this element, so we have

$$(c(A, v_1))\backslash S(\lambda\mu(v_1), \gamma) \leqslant (c(A, v_1))\backslash S(\lambda\mu(v_2), \gamma) \leqslant (c(A, v_2))\backslash S(\lambda\mu(v_2), \gamma). \quad \square$$

If $(x_n)_{n\in\mathbb{N}}$ is a decreasing converging sequence with limit x, we shall write $x_n \underset{n}{\downarrow} x$.

Lemma 42 (Second elimination lemma). *Let $(v_p)_{p\in\mathbb{N}}$ a sequence of words such that (A, v_p) is an increasing sequence with limit s and $\lambda\mu(v_p)$ is a decreasing sequence* *with limit* $\bar{\lambda}$ (*i.e.* $\bar{\lambda} = \bigcap_{p\in\mathbb{N}} \lambda\mu(v_p)$), *then*

$$\bigcap_{p\in\mathbb{N}} S(c(A,v_p),\lambda\mu(v_p),\gamma) = S(cs,\bar{\lambda},\gamma).$$

Proof. Let w be any word. We denote $u_{p,q} = (c(A,v_p))\backslash(\lambda\mu(v_q)\mu(w)\gamma)$. We have

$$u_{p,q} \underset{q}{\downarrow} u_{p,\infty}, \quad \text{where } u_{p,\infty} = (c(A,v_p))\backslash(\bar{\lambda}\mu(w)\gamma) \quad \text{and}$$

$$u_{p,q} \underset{p}{\downarrow} u_{\infty,q}, \quad \text{where } u_{\infty,q} = (cs)\backslash(\lambda\mu(v_q)\mu(w)\gamma).$$

Indeed, for each $k\in\{1,\ldots,n\}$, the hypothesis yields $(\lambda\mu(v_q))_k \underset{q}{\downarrow} \bar{\lambda}_k$, and so for each k and any word w, we have

$$(\lambda\mu(v_q))_k(\mu(w)\gamma)_k \underset{q}{\downarrow} \bar{\lambda}_k(\mu(w)\gamma)_k,$$

that is, summing on k: $\lambda\mu(v_q)\mu(w)\gamma \underset{q}{\downarrow} \bar{\lambda}\mu(w)\gamma$. Since cutting by a constant is like removing it, we conclude that $u_{p,q} \underset{q}{\downarrow} u_{p,\infty}$. For the other convergence, it is even easier.

In the same way, we show that

$$\begin{array}{c} u_{p,\infty} \underset{p}{\downarrow} u_\infty \\ \text{and} \\ u_{\infty,q} \underset{q}{\downarrow} u_\infty, \end{array} \quad \text{where } u_\infty = (cs)\backslash(\bar{\lambda}\mu(w)\gamma).$$

Now, the sequence $(u_{p,p})_{p\in\mathbb{N}}$ decreases, let u'_∞ be its infimum. It is immediate that $u'_\infty \geqslant u_\infty$. Let us show that $u_\infty \geqslant u'_\infty$. Let p and q be two integers such that $p \geqslant q$, we have: $u_{p,q} \geqslant u_{p,p}$, and so $u_{\infty,q} \geqslant u'_\infty$, which leads to $u_\infty \geqslant u'_\infty$. That implies exactly what we wanted to show. □

The rest of the proof relies on properties of well-quasi-ordered sets. In order to keep this paper self-contained, we briefly remind the definition and basic properties of quasi-orders. For more details and proofs, see [7,3]. Recall that a subset D of any ordered set E is an *ideal* if $a\in D$ and $a \leqslant b$ implies that b belongs to D. The *ideal generated* by D, denoted by $\bar{D}$, is the smallest ideal of E containing D, and it is equal to the set of elements of E greater than at least one element of D: $\bar{D} = \{a\in E \mid \exists d\in D,\ d\leqslant a\}$.

Theorem 43 (Higman [3]). *The following conditions on a partially ordered set E are equivalent*:

(1) *every ideal of E is generated by a finite subset*,

(2) *there exists in E neither an infinite strictly descending sequence nor an infinite set of pairwise incomparable elements.*

A quasi-ordered set which satisfies one of the conditions of Theorem 43 is said to be *well-ordered*.

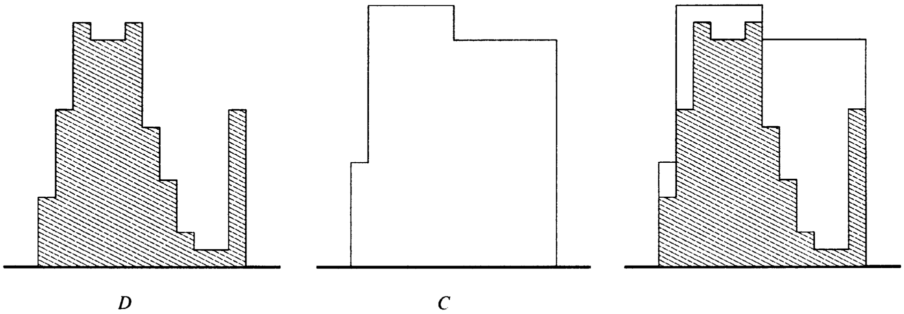

Fig. 2. Examples of contours: D is greater than $C, C \leqslant D$.

Given two ordered sets $(E, \leqslant)$ and $(F, \leqslant)$, their product is the set $E \times F$ ordered by $(e, f) \leqslant (e', f')$ if and only if $e \leqslant e'$ and $f \leqslant f'$.

Proposition 44. *If two sets are well ordered, so is their product.*

Corollary 45 (Dickson). *For any integer r, the ordered set $(N^r_{\min}, \geqslant)$ is well-ordered.*

Proof. The set $(\mathbb{N}_{\min}, \geqslant)$ is well-ordered. Indeed, if C is an ideal of $(\mathbb{N}_{\min}, \geqslant), C$ admits a $\leqslant$-maximal element, say c. And by definition, we have $C = \bar{c}$. So $(\mathbb{N}_{\min}, \geqslant)$ is a well-ordered set and by Proposition 44, so is $(\mathbb{N}^r_{\min}, \geqslant)$. □

We call *r-contour* (or simply *contour*) an element of $\mathbb{N}^r_{\min}$. A graphical representation is given in Fig. 2 (the contour D represents the vector $(4, 9, 14, 13, 13, 14, 9, 5, 2, 1, 1, 9), r = 12$). Contours are partially ordered by the product order on $\mathbb{N}^r_{\min}$.

If C is a subset of an ordered set $(E, \leqslant)$, a decreasing sequence of elements of C is a sequence $(x_n)_{n \geqslant 0}$ such that $x_n \geqslant x_{n+1}$ for all n. This includes constant sequences. If $(E, \leqslant) = (\mathbb{N}_{\min}, \leqslant)$, the limit of a decreasing sequence is its infimum.

Let $\mathscr{C}$ be a set of r-contours. We denote by $\mathscr{C}^{\downarrow}$ the set of r-contours containing $\mathscr{C}$ and all the limits of decreasing sequences of elements of $\mathscr{C}$.

Lemma 46. *Let $\mathscr{C}$ be a set of r-contours. There exists a finite set $\mathscr{D}$ of r-contours such that*

(1) *each element of $\mathscr{D}$ is the limit of a decreasing sequence of elements of $\mathscr{C}$,*
(2) *for each $C \in \mathscr{C}$, there exists $D \in \mathscr{D}$ such that $D \leqslant C$.*

Proof. We let $\mathscr{D}$ be the set of minimal elements of $\mathscr{C}^{\downarrow}$. By definition, the pairwises of elements of $\mathscr{D}$ are incomparable, so by Theorem 43 and Corollary 45, the set $\mathscr{D}$ is finite. Furthermore, $\mathscr{D}$ satisfies condition 1 because it is a subset of $\mathscr{C}^{\downarrow}$ and condition 2 because $\mathscr{C}^{\downarrow}$ contains all the limits of its decreasing sequences. □

We can now return to Theorem 40.

Proof of Theorem 40. We now analyze the case where the acceptance formula of X is an atom (c,λ,γ): $X=S(c,\lambda,\gamma)$. Formula (8) becomes then

$$A\backslash X = \bigcap_{v\in\Sigma^*} S(c(A,v),\lambda\mu(v),\gamma).$$

We can rewrite this formula, by separating the coefficients of A:

$$\begin{aligned} A\backslash X &= \bigcap_{s\in\mathbb{N}_{\min}} \bigcap_{v\in A^{-1}s} S(cs,\lambda\mu(v),\gamma) \\ &= \bigcap_{s\in\operatorname{Im}A} \bigcap_{v\in A^{-1}s} S(cs,\lambda\mu(v),\gamma). \end{aligned}$$

For each $s\in\operatorname{Im}A$, we consider the set of contours $\mathscr{C}_s=((\lambda\mu(v))_{v\in A^{-1}s}$ and we apply Lemma 46. We obtain a finite set $\mathscr{D}_s$ of contours such that $\mathscr{D}_s\subseteq\mathscr{C}_s^{\downarrow}$ (Condition 1 of Lemma 46), thus

$$\bigcap_{\bar\lambda\in\mathscr{C}_s^{\downarrow}} S(cs,\bar\lambda,\gamma) \leqslant \bigcap_{\bar\lambda\in\mathscr{D}_s} S(cs,\bar\lambda,\gamma),$$

by Condition 2 of Lemma 46, there exists a subset $\mathscr{D}'_s$ of $\mathscr{D}_s$ such that

$$\bigcap_{\bar\lambda\in\mathscr{D}'_s} S(cs,\bar\lambda,\gamma) \leqslant \bigcap_{\bar\lambda\in\mathscr{C}_s} S(cs,\bar\lambda,\gamma).$$

But, by definition of $\mathscr{C}_s^{\downarrow}$, we have $\bigcap_{\bar\lambda\in\mathscr{C}_s^{\downarrow}} S(cs,\bar\lambda,\gamma)=\bigcap_{\bar\lambda\in\mathscr{C}_s} S(cs,\bar\lambda,\gamma)$ and $\mathscr{D}'_s\subseteq\mathscr{D}_s$ implies that $\bigcap_{\bar\lambda\in\mathscr{D}_s} S(cs,\bar\lambda,\gamma)\leqslant\bigcap_{\bar\lambda\in\mathscr{D}'_s} S(cs,\bar\lambda,\gamma)$. We can conclude that

$$\bigcap_{\bar\lambda\in\mathscr{C}_s} S(cs,\bar\lambda,\gamma) = \bigcap_{\bar\lambda\in\mathscr{D}_s} S(cs,\bar\lambda,\gamma).$$

Since the image of A is finite, $\bigwedge_{s\in\operatorname{Im}A}\bigwedge_{\lambda\in\mathscr{D}_s}(cs,\bar\lambda,\gamma)$ is an acceptation formula for the series $A\backslash X$.

Let us consider the case where X is any pseudo-recognizable series. We know, according to Proposition 39, that X admits a multi-representation with a unique final vector, a unique coefficient and a purely conjunctive formula. Let $\Phi_1=(c,\lambda_1,\gamma)$ and $\Phi_2=(c,\lambda_2,\gamma)$ be atoms which appear in the acceptance formula of X, and let $X_1=S(\Phi_1)$ and $X_2=S(\Phi_2)$. By Lemma 17, we have $A\backslash(X_1\cap X_2)=(A\backslash X_1)\cap(A\backslash X_2)$ and so $A\backslash(X_1\cap X_2)$ 0 is recognized by a multi-representation with the same base, the same unique final vector, the same unique coefficient as X and without any disjunction.

By iteration on the number of atoms of the acceptance formula of X, we obtain the result for any pseudo-recognizable X. □

It is now immediate that each K_n is pseudo-recognizable, and they all admit multi-representations with the same base and the same unique final vector.

6.2.2. The solution to our problem

We now show that the limit K_∞, i.e. the supremum of the set $E=\{X\mid AX\leqslant X+B$ and $X\leqslant K\}$, is also pseudo-recognizable, when the series A is a language.

Theorem 47. *Let $(L_n)_{n\in\mathbb{N}}$ be a sequence of pseudo-recognizable series with the same base, the same unique final vector and the same unique coefficient. Then the series $\bigcap_{n\in\mathbb{N}} L_n$ is pseudo-recognizable.*

Proof. By Proposition 39, the series L_n admit multi-representations with purely conjunctive formulas, and with the same base, the same unique final vector and the same unique coefficient. Let γ be the unique final vector and c be the unique coefficient of the acceptance formulas of the series L_n. We consider the acceptance formulas of the series L_n without disjunction, and we apply Lemma 46 to the set of contours λ, such that (c,λ,γ) occurs in the acceptance formula of some L_n. We obtain a finite set $\mathscr{D}$ of contours such that $\bigwedge_{\lambda\in\mathscr{D}}(c,\lambda,\gamma)$ is an acceptance formula for $\bigcap_n L_n$. $\square$

Let us show that the sequence $(K_n)_{n\in\mathbb{N}}$ satisfies the hypothesis of Theorem 47, when A is a language. Series B and K are pseudo-recognizable and are recognized by multi-representations with a unique final vector (respectively, γ_B and γ_K), a unique coefficient, and without any disjunction according to Proposition 39. Making the same construction as in the proof of Lemma 24, but with the final vector (γ_B/γ_K), we obtain multi-representations for B and K with same base μ, same unique final vector γ and same unique coefficient c. According to Proposition 39, they have a multi-representation which moreover does not have disjunction, keeping same base, final vector and coefficient. We say that a series is *adequate* if it has a multi-representation with base μ, unique final vector γ, unique coefficient c and without disjunction. The series B and K are adequate, and so is the series $B+K$. By Theorem 40, we know that $A\backslash(B+K)$ is adequate, since A is a language. Therefore, K_1 is adequate. By iteration, all K_n are adequate. Now we apply Theorem 47, and we conclude that $K_\infty=\bigcap_n K_n$ is a pseudo-recognizable series.

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References

[1] J. Berstel, C. Reutenauer, Rational Series and their Languages, Springer, Berlin, 1988.

[2] J. Gunawardena, An introduction to idempotency, in: J. Gunawardena (Ed.), Idempotency, Cambridge University Press, Cambridge, 1998 (Chapter 1).

[3] G. Higman, Ordering by divisibility in abstract algebras, Proc. London Math. Soc. 2 (1952) 326–336.

[4] I. Klimann, New types of automata to solve fixed point problems, Theoret. Comput. Sci. 259 (2001) 183–197.

[5] J.-E. Pin, Syntactic semigroups, in: G. Rozenberg, A. Salomaa (Eds.), Handbook of Language Theory, Springer, Berlin, 1997 (Chapter 10).

[6] J.-E. Pin, J. Sakarovitch, Une application de la représentation matricielle des transductions, Theoret. Comput. Sci. 35 (1985) 271–293.
[7] J. Sakarovitch, I. Simon, Subwords, in: M. Lothaire (Ed.), Combinatorics on Words, Addison-Wesley, Reading, MA, 1983 (Chapter 6).
[8] W.M. Wonham, Linear Multivariable Control—A Geometric Approach, 3rd Edition, Springer, Berlin, 1985.